

Near Neighbor Problem Made Fair

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Nearest Neighbor Problems

- **Nearest Neighbor:** Given a set of objects, find the closest one to the query object.



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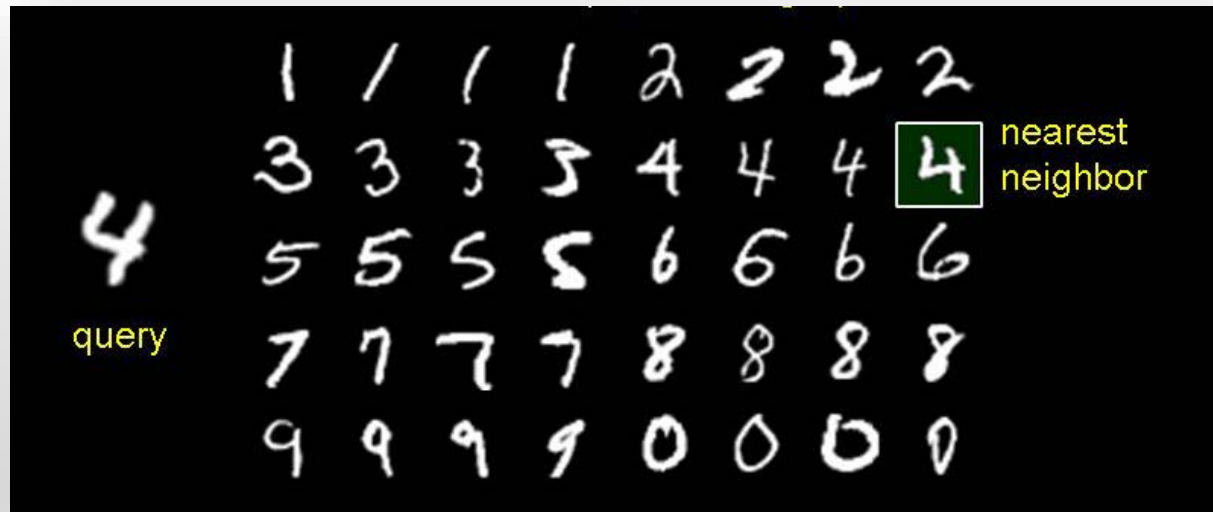
Nearest Neighbor Problems

- **Nearest Neighbor:** Given a set of objects, find the closest one to the query object.
- **Near Neighbor:** given a set of objects, find one that is close enough to the query object.



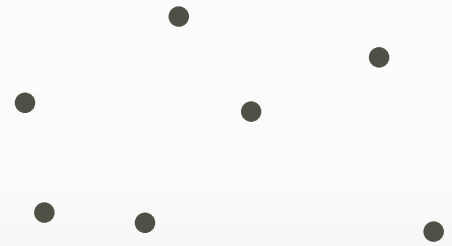
There are many applications of NN

Searching for the closest object



Near Neighbor

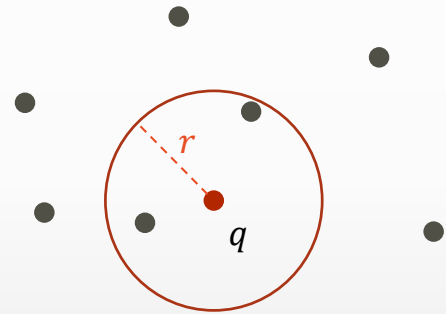
Dataset of n points P in a metric space, e.g. $\mathbb{R}^d$,
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A query point q comes online



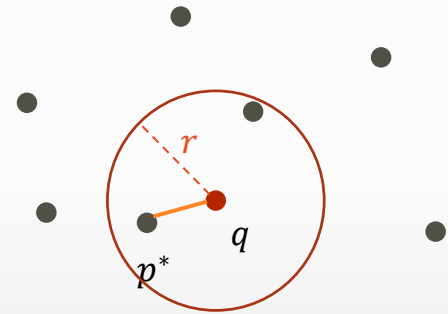
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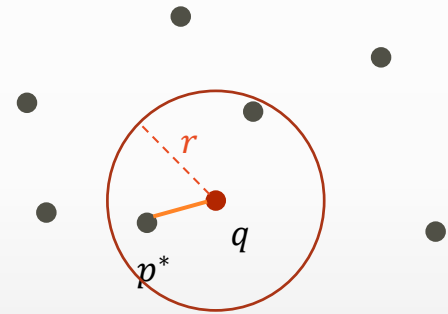
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Goal:

- Find a point p^* in the r -neighborhood
- Do it in sub-linear time and small space



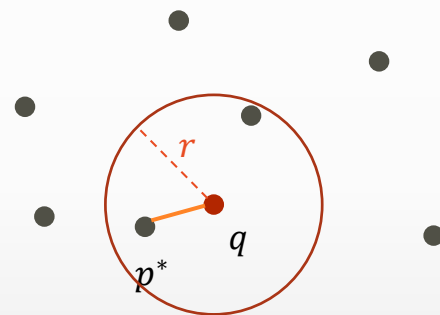
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All existing algorithms for this problem

- Either space or query time depending exponentially on d
- Or assume certain properties about the data, e.g., bounded intrinsic dimension

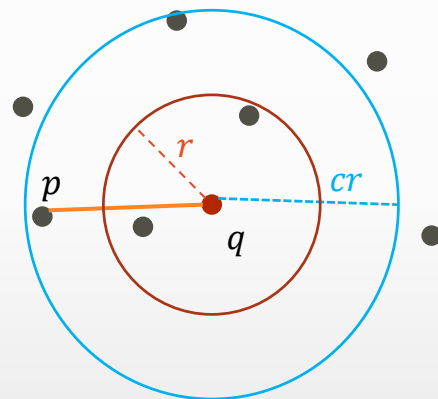
Approximate Near Neighbor

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 - Report a point in distance cr for $c > 1$



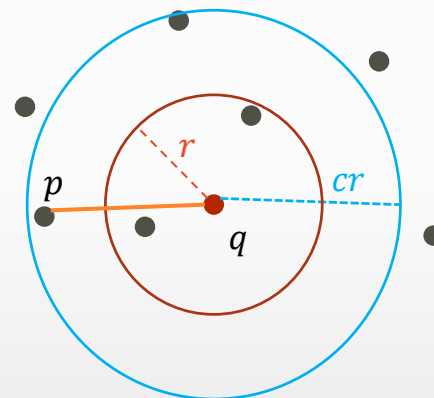
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- **Approximate Near Neighbor**
 - Report a point in distance cr for $c > 1$
 - For **Hamming** (and **Manhattan**) query time is $n^{O(1/c)}$ [IM98]
 - and for **Euclidean** it is $n^{O(\frac{1}{c^2})}$ [AI08]



Fair Near Neighbor

Sample a neighbor of the query **uniformly at random**

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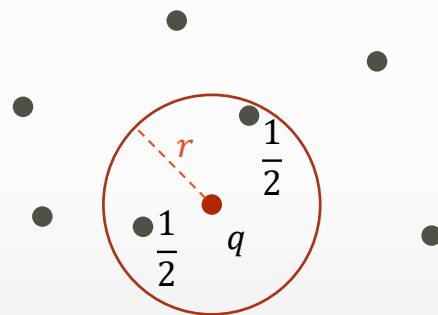
Applications:

- ☐ Removing noise, k-NN classification
- ☐ Anonymizing the data
- ☐ Counting the neighborhood size

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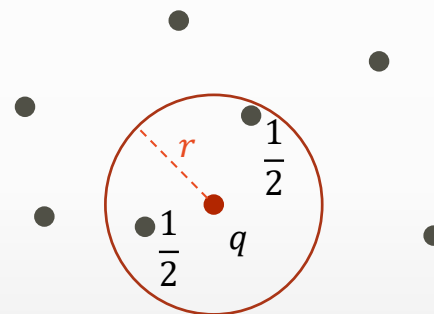
Goal:

- Return each point p in the neighborhood of q with uniform probability
- Do it in sub-linear time and small space

Approximate Fair Near Neighbor

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Goal of **Approximate Fair NN**

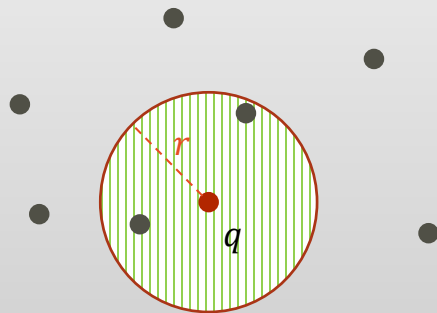
- Any point p in $N(q, r)$ is reported with “almost uniform” probability, i.e., $\lambda_q(p)$ where

$$\frac{1}{(1 + \epsilon)|N(q, r)|} \leq \lambda_q(p) \leq \frac{(1 + \epsilon)}{|N(q, r)|}$$

Results on $(1 + \epsilon)$ -Approximate Fair NN

Domain	Guarantee	Space	Query
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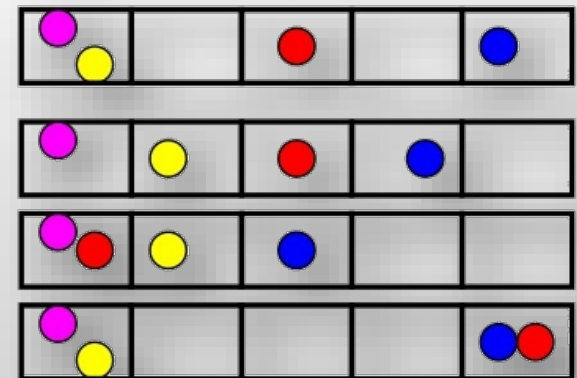
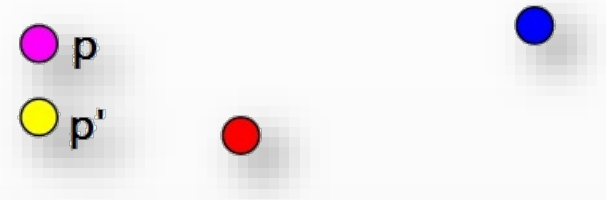
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- **Experiments**
- Recent paper [Aumuller, Pagh, Silvestry'19] defining the same notion

Locality Sensitive Hashing (LSH)

One of the main approaches to solve the Nearest Neighbor problems

Locality Sensitive Hashing (LSH)

Hashing scheme s.t. close points have higher probability of collision than far points

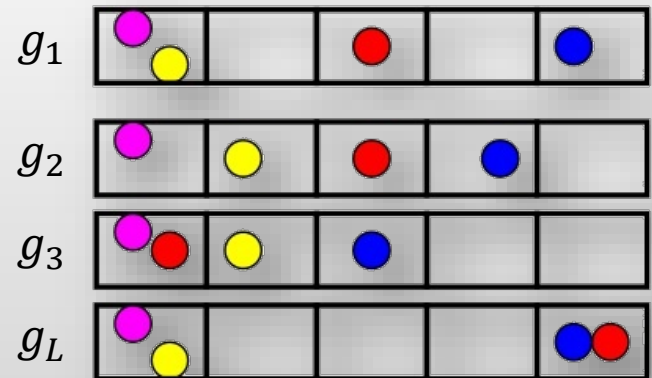
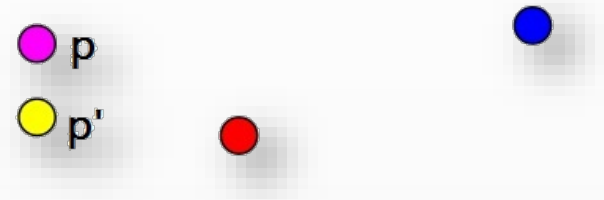


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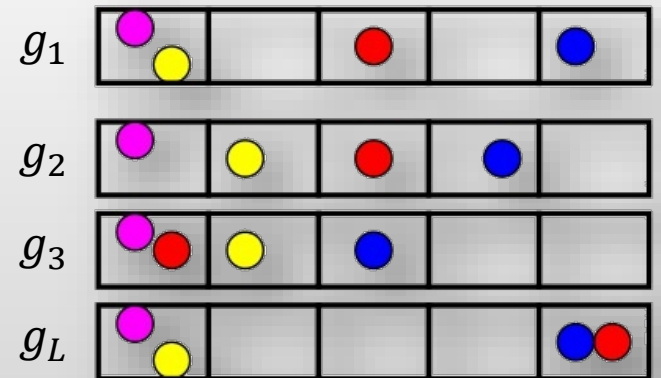
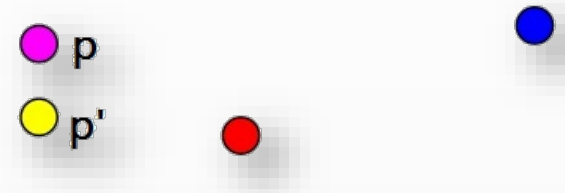
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If $\|p - p'\| \leq r$, they collide w.p. $\geq P_{high}$

If $\|p - p'\| \geq cr$, they collide w.p. $\leq P_{low}$

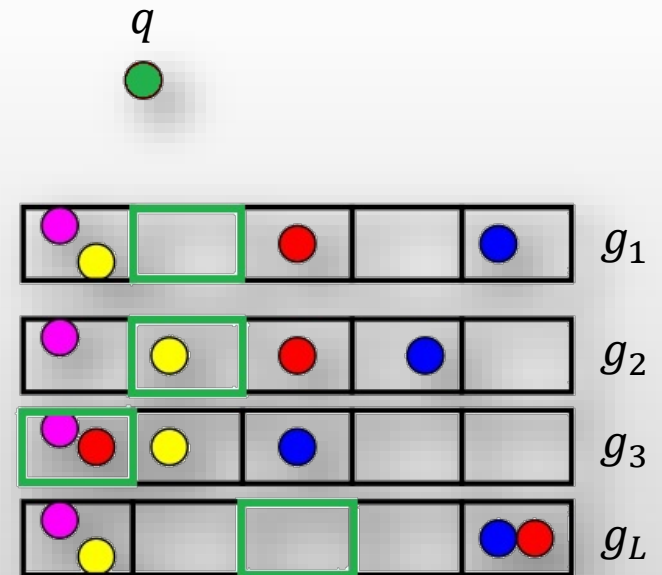
For $P_{high} \geq P_{low}$



Locality Sensitive Hashing (LSH)

Retrieval: [Indyk, Motwani'98]

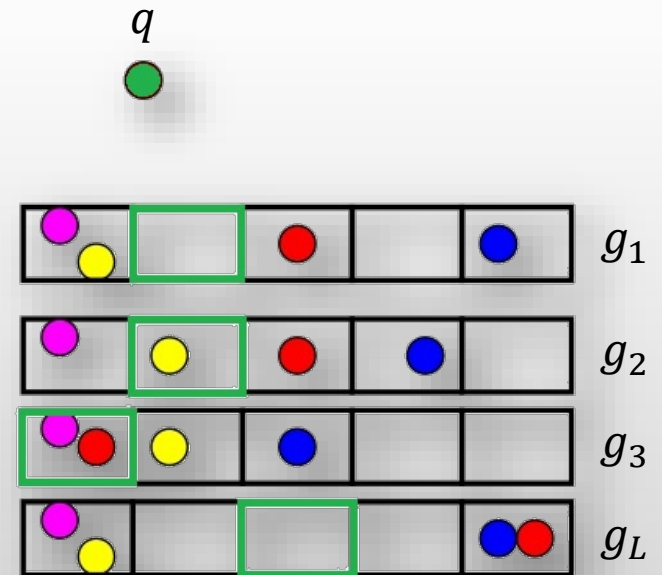
- The **union of the query buckets** is roughly the neighborhood of q
- $\bigcup_i B_i(g_{i(q)})$ is roughly the neighborhood



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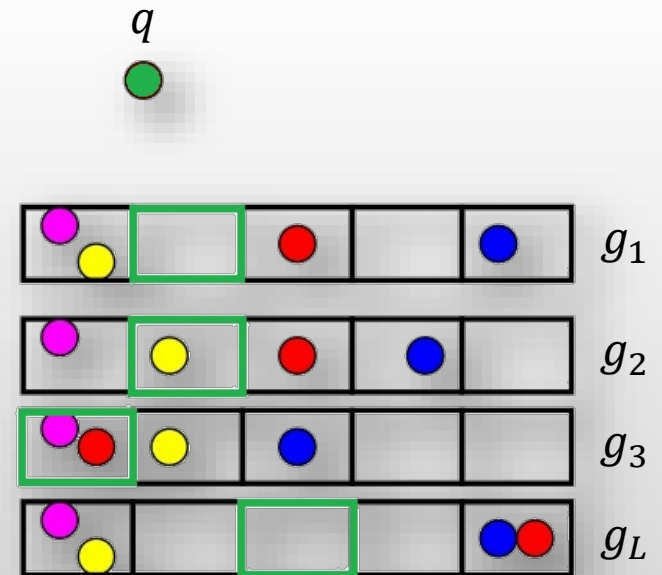
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- How to report a **uniformly random** neighbor from **union** of these buckets?
- Collecting all points might take $O(n)$ time



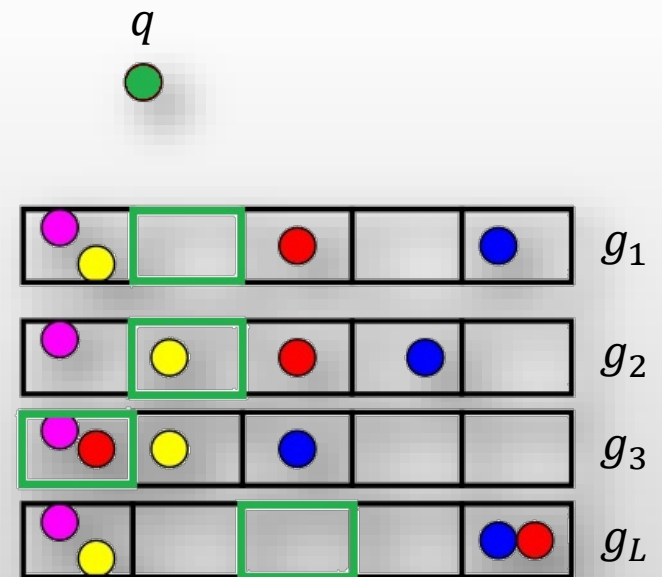
Approaches

Approach 1: Uniform/Uniform

How to output a random neighbor from

$U_i B_i(g_{i(q)})$:

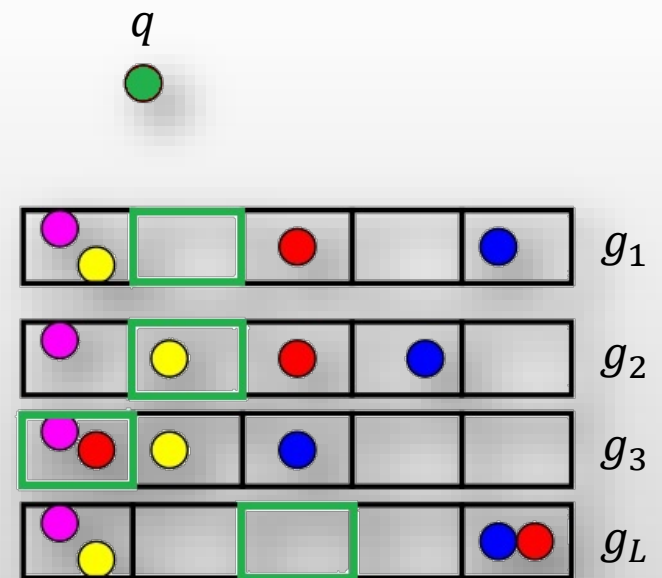
1. Choose a uniformly random **bucket**
2. Choose a uniformly random **point** in the bucket



Approach 2: Weighted/Uniform

How to output a random neighbor from $U_i B_i(g_{i(q)})$:

1. Choose a random bucket proportional to its size
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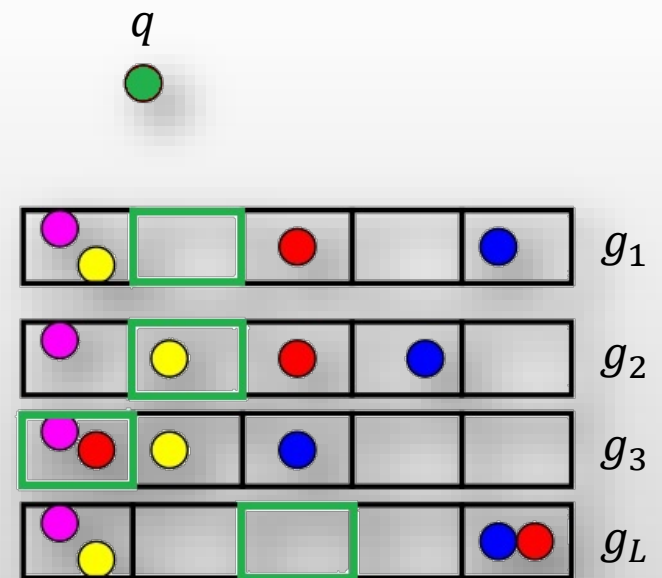


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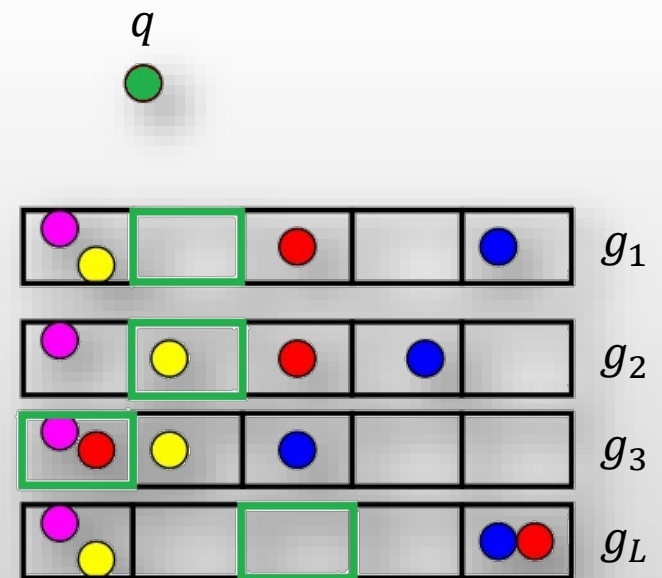
Number of buckets that p appears in



Approach 3: Optimal

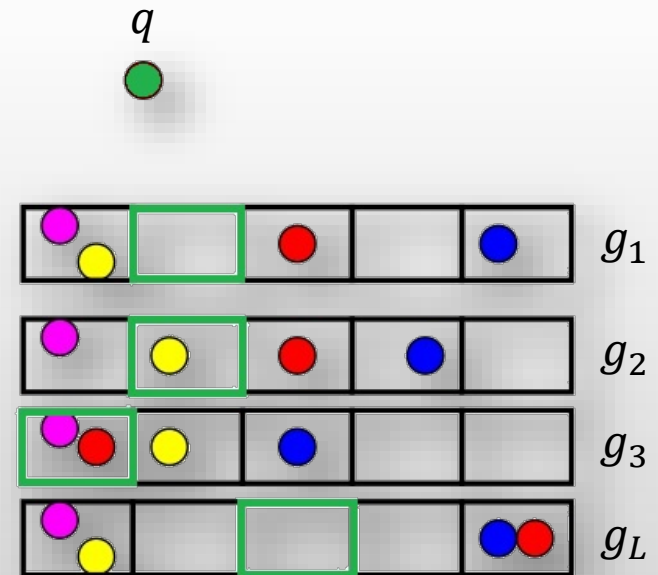
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3. **Keep** p with probability $\frac{1}{d_p}$, o.w. **repeat**



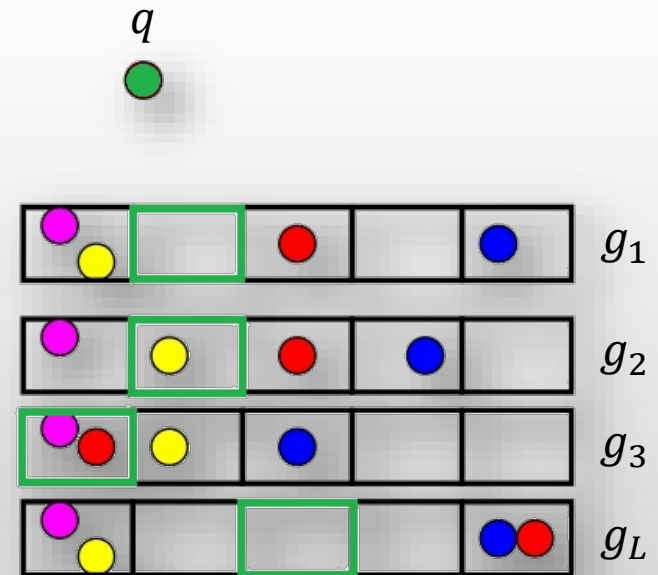
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 - Uniform probability



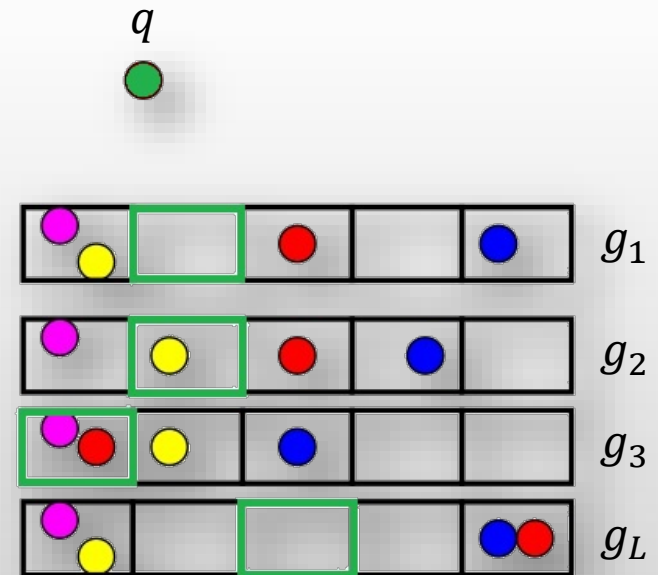
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 - Uniform probability
 - Need to spend $O(L)$ to find the degree
 - Might need $O(d_{max}) = O(L)$ samples
 - Total time is $O(L^2)$



Approximate the degree d_p

Sample $O(\frac{L}{d_p \cdot \epsilon^2})$ buckets out of L buckets to $(1 + \epsilon)$ -approximate the degree.

➤ Still if the degree is low this takes $O(L)$ samples.

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Case 1: Small degree d_p :

- More samples are required to estimate
- Reject with lower probability -> Fewer queries of this type

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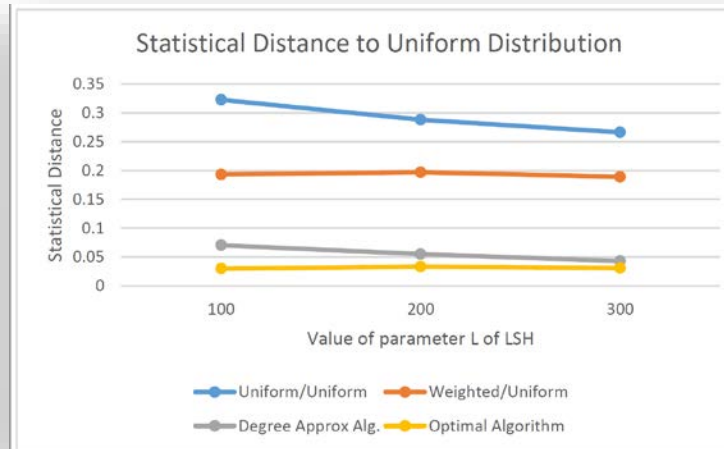
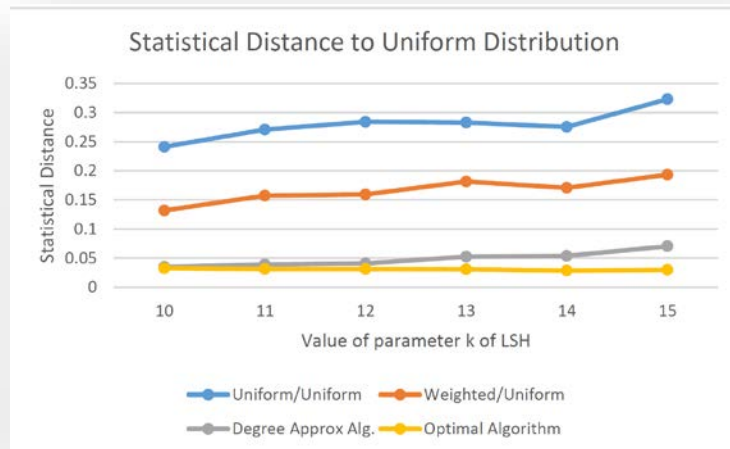
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➤ This decreases $O(L^2)$ runtime to $\tilde{O}(L)$

➤ Large dependency on ϵ of the form $O(\frac{1}{\epsilon^2})$

➤ Via a different sampling approach we show how to reduce the dependency to logarithmic $O(\log \frac{1}{\epsilon})$.

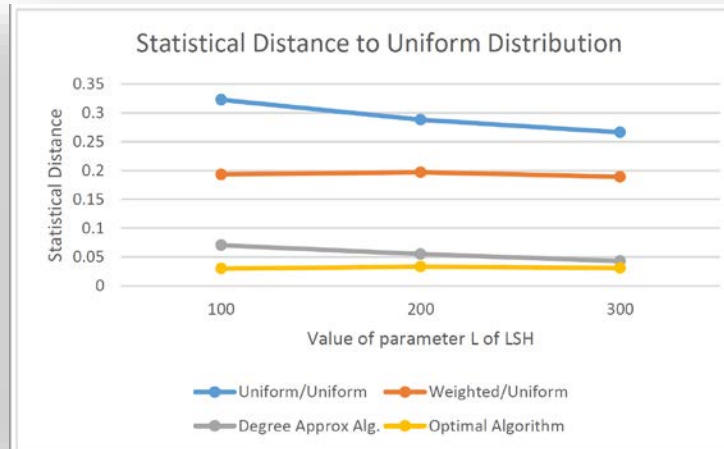
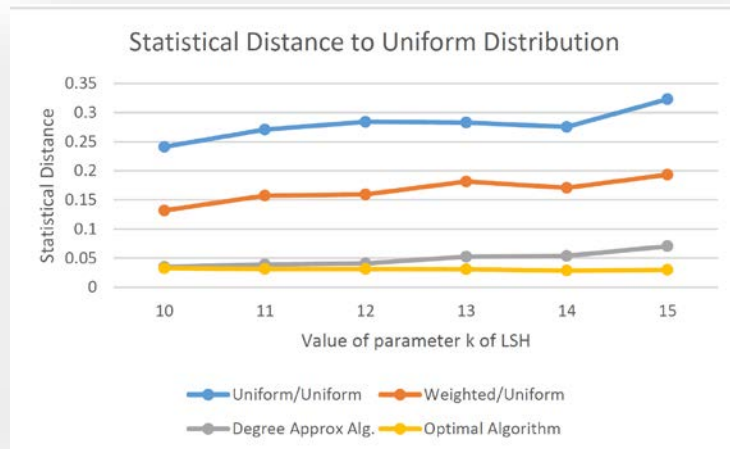
Experiments



Setup

- Take MNIST as the data set
- Ask a query several times and compute the empirical distribution of the neighbors.
- Compute the statistical distance of the empirical distribution to the uniform distribution

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Comparison

- Our algorithm performs 2.5 times worse than the optimal algorithm, but the other two perform 7 and 10 times worse than the optimal.
- Four times faster than the optimal but 15 times slower than the other two

Conclusion

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- ✓ We get an independent near neighbor each time we draw a sample.
- ✓ More generally the approach works for sampling from a sub-collection of sets.

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Thanks
Questions?

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